A Degree of Bounded Mappings Between *M*-Fuzzifying Bornological Space

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Abstract. In this paper, we define the degree of bounded mappings between *M*-fuzzifying bornological spaces. This concept describes a mapping between *M*-fuzzifying bornological spaces to be a bounded mapping in some degree. Also, we generalize some properties with respect to bounded mappings in classical bornological spaces to the fuzzy setting.

Keywords: M-fuzzifying bornological space; the degree of bounded mapping.

1. Introduction

In order to apply the conception of boundedness to the case of a general topological space, S.T. Hu [10,11] first introduced an axiomatic approach to bornology. General bornological spaces play a important role in research of topologies on function spaces [4,14], in optimization theory [5] and in study of convergence in hyperspaces [2,3,13]. Since Zadeh [21] introduced the concept of fuzzy sets, fuzzy set theory has been applied to various branches of mathematics, such as fuzzy control, fuzzy topology, fuzzy algebra and so on. In 2011, Abel and $\check{S}ostak$ [1] firstly generalized the notion of axiomatic bornology to the fuzzy case, which is called L-bornology. Each L-bornology on a set X is just an ideal in the lattice L^X satisfying $\bigvee_{B\in B} B(x)=T_{\mathrm{L}}, \forall x\in X$. In the setting of *L*-bornology, Paseka et al. [17] introduced the categories of *L*-bornological vector spaces and Zhang and Zhang [20] proposed the concept of *I*-bornological vector spaces.

In a different way, $\check{S}ostak$ and Uljane [19] introduced a new approach to the fuzzification of bornology, which is called (L,∗)-valued bornology. Different from *L*-bornology, each (L,∗)-valued bornology on a set *X* is a mapping from 2^X (the powerset of *X*) to *L* satisfying *L*-valued analogues of the axioms of a bornology. In the situation of *L*-valued bornology, $\check{S}ostak$ and Uljane [19] proposed *L*-valued bornologies induced by fuzzy metrics and relative compactness type *L*-valued bornologies in Chang-Goguen *L*-topological spaces. For convenience, we call this fuzzy bornology an fuzzifying bornology.

It is well known that one main character of fuzzy set theory is to equip a mathematical object with some degree and give the related conclusions a degree representation. In [15] defined degrees of Ti (i = 0, 1, 2) separation property as well as regular property of stratified *L*-generalized convergence spaces and investigated their relationship. Recently, Pang [16] applied the fuzzy inclusion degrees between stratified *L*-filters to define stratified *L*-ordered filter spaces and investigated its categorical properties.

The structure of this paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we introduce the definition of degree of bounded mappings between *M*-fuzzifying bornological space, and will give a degree representation to the related conclusions in *M*-fuzzifying bornological spaces.

2. Preliminaries

We consider in this paper complete lattices *L* and *M* where finite meets distributive over arbitrary joins, i.e., $a\wedge\bigvee_{i\in I} b_i=\bigvee_{i\in I}(a\wedge b_i)$ holds for all $a, b_i (i\in I)$. These lattices are called complete

Heyting algebras (or frames). The bottom and top element of *L* (resp. *M*) is denoted by $\perp_L$ and T_L (resp. $\perp_M$ and T_M). Obviously, each completely distributive lattice is a complete Heyting algebra.

Throughout this paper, *X* always denotes a universe of discourse. 2^X and $F(X)$ denote the classes of all crisp and fuzzy subsets of *X*, respectively. The notation 2^X is the set of all non-empty finite subsets of *X*. For any mapping $F: X \to Y$, the notation $F^{\to}: 2^X \to 2^Y$ is defined by $F^{\to}(U) = \{F(x) \in Y \mid x \in U\}$ for $U \in 2^X$ and $F^{\leftarrow}: 2^Y \to 2^X$ is defined by $F^{\leftarrow}(V) = \{x \in X \mid F(x) \in V\}$ for $V \in 2^X$, respectively (see[6][18]). We define the residual implication as: $a \to b = \bigvee\{\lambda \in M \mid a \wedge \lambda \le b\}$. Also, we define $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. We will then often use the following properties implied by residual implication.

2.1 Lemma [8] Let *M* be a frame. Then the following statements holds:

(H1) $T \to a = a$;
(H2) $a \le b$ if and only if $a \to b = T$;
(H3) $(a \to b) \to b \ge a$;
(H4) $(a \to b) \wedge (b \to c) \le a \to c$;
(H5) $a \to \bigwedge_{j \in J} b_j = \bigwedge_{j \in J}(a \to b_j)$;
(H6) $\bigvee_{j \in J} a_j \to b = \bigwedge_{j \in J}(a_j \to b)$;
(H7) $\bigwedge_{j \in J} a_j \to \bigwedge_{j \in J} b_j \ge \bigwedge_{j \in J}(a_j \to b_j)$;
(H8) $\bigvee_{j \in J} a_j \to \bigvee_{j \in J} b_j \ge \bigwedge_{j \in J}(a_j \to b_j)$;
(H9) $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \le a \leftrightarrow c$.
Clearly, every completely distributive lattice is a frame.

2.2 Lemma [19] An *M*-fuzzifying bornology on a set *X* is a mapping $\mathrm{B}: 2^X \to M$ satisfying the following conditions:

(MB1) $\mathrm{B}(\{x\}) = T_M, \forall x \in X$;

(MB2) For all $A, B \in 2^X$ with $A \subseteq B, \mathrm{B}(A) \ge \mathrm{B}(B)$;

(MB3) $\mathrm{B}(A \cup B) \ge \mathrm{B}(A) \wedge \mathrm{B}(B), \forall A, B \in 2^X$.
The pair is called an *M*-fuzzifying bornological space. The value $\mathrm{B}(A)$ is interpreted as the degree of boundedness of a set *A* in the space (X, B).

2.3 Definition [9] A mapping $f: (X, \mathrm{B}_X) \to (Y, \mathrm{B}_Y)$ between two *M*-fuzzifying bornological spaces is called *M*-fuzzifying bounded-preserving if $\mathrm{B}_X(f^{-1}(B)) \ge \mathrm{B}_Y(B)$ for all $B \in 2^Y$.

2.4 Definition [19] A mapping $f: (X, \mathrm{B}_X) \to (Y, \mathrm{B}_Y)$ between two *M*-fuzzifying bornological spaces is called *M*-fuzzifying bounded if $\mathrm{B}_X(A) \le \mathrm{B}_Y(f(A))$ for all $A \in 2^X$.

2.5 Proposition [19] If mapping $f: (X, \mathrm{B}_X) \to (Y, \mathrm{B}_Y)$ and $g: (Y, \mathrm{B}_Y) \to (Z, \mathrm{B}_Z)$ are *M*-fuzzifying bounded, then $g \circ f: (X, \mathrm{B}_X) \to (Z, \mathrm{B}_Z)$ is *M*-fuzzifying bounded.

2.6 Proposition [12] Let (X, B) be an *M*-fuzzifying bornological space. For any $Y \in 2^X$, define $\mathrm{B}\|_Y: 2^Y \to M$ by

$$\mathrm{B}\|_Y(U) = \bigvee_{V \in 2^X, V|_Y = U} \mathrm{B}(V), \forall U \in 2^Y.$$

Then $(Y, \mathrm{B}|_Y)$ is an *M*-fuzzifying bornological space, which is called the subspace of (X, B).

2.7 Definition [7] Let *X* be a vector space over …and $A \in 2^X$. We say that *A* is circled if $tA \le A$ for any $t \in \ldots$ with $|t| \le 1$. The set of circled sets is denoted by $Cir(X, \ldots)$.

2.8 Definition [12] Let (X, B) be an *M*-fuzzifying bornological linear space and let $\{x_n\}$ be a sequence in X. The degree to which x_n is convergent to x bornologically is

$$\left[x_n \xrightarrow{M} x\right] = \bigvee_{\substack{A \in Cir(X, \ldots) \\ \lambda_n \to 0}} \left\{\mathrm{B}_X(A) : x^n - x \in \lambda_n A, \forall n \in N\right\}.$$

3. The degree of bounded mappings between M-fuzzifying bornological spaces

The aim of this section is to introduce the concept of the degree of bounded mappings between *M*-fuzzifying bornological spaces.

3.1 Definition Let (X, B_X) and (X, B_Y) be two *M*-fuzzifying bornological spaces, $f : X \to Y$ be a mapping. Then the degree $DB(f)$ to which *f* is bounded is defined as follows

$$DB(f) = \bigwedge_{A \in 2^X} (\mathrm{B}_X(A) \to \mathrm{B}_Y(f(A))).$$

3.2 Remark If $DB(f) = T_M$, by Definition 2.4, we know $\mathrm{B}_X(A) \le \mathrm{B}_Y(f(A))$ for all $A \in 2^X$. This is exactly the definition of bounded mappings in *M*-fuzzifying bornological space.

3.3 Proposition Let $f : (X, \mathrm{B}_X) \to (Y, \mathrm{B}_Y)$ and $g : (Y, \mathrm{B}_Y) \to (Z, \mathrm{B}_Z)$ be mappings between *M*-fuzzifying bornological spaces. Then

$$DB(f) \wedge DB(g) \le DB(g \circ f).$$

Proof. By Definition 3.1, we have

$$\begin{aligned}
& DB(f) \wedge DB(g) \\
= & \bigwedge_{A \in 2^X} \left(\mathrm{B}_X(A) \to \mathrm{B}_Y(f(A))\right) \wedge \bigwedge_{B \in 2^Y} \left(\mathrm{B}_Y(B) \to \mathrm{B}_Z(g(B))\right) \\
\le & \bigwedge_{A \in 2^X} \left(\mathrm{B}_X(A) \to \mathrm{B}_Y(f(A))\right) \wedge \bigwedge_{C \in 2^X} \left(\mathrm{B}_Y(f(C)) \to \mathrm{B}_Z(g(f(C)))\right) \\
= & \bigwedge_{A \in 2^X} \left(\mathrm{B}_X(A) \to \mathrm{B}_Y(f(A))\right) \wedge \left(\mathrm{B}_Y(f(A)) \to \mathrm{B}_Z((g \circ f)(A))\right) \\
\le & \bigwedge_{A \in 2^X} \left(\mathrm{B}_X(A) \to \mathrm{B}_Z((g \circ f)(A))\right) \\
= & DB(g \circ f).
\end{aligned}$$

3.4 Corollary Let $f : (X, \mathrm{B}_X) \to (Y, \mathrm{B}_Y)$ and $g : (Y, \mathrm{B}_Y) \to (Z, \mathrm{B}_Z)$ be mappings between *M*-fuzzifying bornological spaces.

(1) If *f* is surjective, then $DB(g \circ f) \wedge DB(f^{-1}) \le DB(g)$;

(2) If *f* is surjective, then $DB(g \circ f) \wedge DB(g^{-1}) \le DB(f)$;

3.5 Proposition Let (X, B) be an *M*-fuzzifying bornological space, $(Y, \mathrm{B}|_Y)$ be the subspace of (X, B), and $f : Y \to X$ be the including mapping. Then $DB(f) = T_M$.

Proof. By Definition 2.6, we have

$$\begin{aligned} DB(f) &= \bigwedge_{A\in 2^Y}(\mathrm{B}|_Y(A)\to \mathrm{B}(A)) \\ &= \bigwedge_{A\in 2^Y}\left(\left(\bigvee_{B\in 2^X, B|_Y=A}\mathrm{B}(B)\right)\to \mathrm{B}(A)\right) \\ &= \bigwedge_{A\in 2^Y}\ \bigwedge_{B\in 2^X, B|_Y=A}\mathrm{B}(B)\to \mathrm{B}(A) \\ &= T_M. \end{aligned}$$

3.6 Lemma let X and Y be vector spaces over …, $f:Y\to X$ be a linear mapping. Then

(1) $f(tA)=tf(A), \forall t\in \ldots$;

(2) If $A\in Cir(X,\ldots)$), then $f(A)\in Cir(Y,\ldots)$.

3.7 Proposition Let (X,B_X) and (X,B_Y) be two *M*-fuzzifying bornological spaces, $f:X\to Y$ be a linear mapping. Then

$$DB(f)\le \bigwedge_{\{x_n\}\subseteq X}\left([x_n \xrightarrow{M} x]\to [f(x_n)\xrightarrow{M} f(x)]\right).$$

Proof. Take any $A\in Cir(X,\ldots)$ and $\lambda_n\to 0$ such that $x_n\to x\in \lambda_n A$. Since $f:X\to Y$ is a linear mapping, it follows that

$$f(x_n)-f(x)=f(x_n-x)\in f(\lambda_n A)=\lambda_n f(A).$$

By Lemma 3.6, we know $f(A)\in Cir(Y,\ldots)$. Then

$$\begin{aligned} &\bigwedge_{\{x_n\}\subseteq X}\left([x_n \xrightarrow{M} x]\to [f(x_n)\xrightarrow{M} f(x)]\right) \\ &= \bigwedge_{\{x_n\}\subseteq X}\left(\begin{array}{l}\bigvee_{A\in Cir(X,\ldots),\ \lambda_n\to 0}\left\{\mathrm{B}_X(A): x^n-x\in \lambda_n A, \forall n\in N\right\}\to \\ \bigvee_{B\in Cir(Y,\ldots),\ \lambda_n\to 0}\left\{\mathrm{B}_Y(B): f(x^n)-f(x)\in \lambda_n B, \forall n\in N\right\}\end{array}\right) \\ &\ge \bigwedge_{\{x_n\}\subseteq X}\left(\begin{array}{l}\bigvee_{A\in Cir(X,\ldots),\ \lambda_n\to 0}\left\{\mathrm{B}_X(A): x^n-x\in \lambda_n A, \forall n\in N\right\}\to \\ \bigvee_{f(A)\in Cir(Y,\ldots),\ \lambda_n\to 0}\left\{\mathrm{B}_Y(f(A)): f(x^n)-f(x)\in \lambda_n f(A), \forall n\in N\right\}\end{array}\right) \\ &= \bigwedge_{\{x_n\}\subseteq X}\ \bigwedge_{A\in Cir(X,\ldots),\ \lambda_n\to 0}\left\{\mathrm{B}_X(A)\to \mathrm{B}_Y(f(A)): x^n-x\in \lambda_n A, \forall n\in N\right\} \\ &\ge \bigwedge_{A\in 2^X}\left(\mathrm{B}_X(A)\to \mathrm{B}_Y(f(A))\right) \\ &= DB(f). \end{aligned}$$

4. The Degree of Bounded-Preserving Mappings Between *M*-Fuzzifying Bornological Spaces

The aim of this section is to introduce the concept of the degree of bounded-preserving mappings between *M*-fuzzifying bornological spaces.

4.1 Definition Let (X,B_X) and (X,B_Y) be two *M*-fuzzifying bornological spaces, $f:X\to Y$ be a linear mapping. Then the degree $DP(f)$ to which f is bounded-preserving is defined as follows

$$BP(f)=\bigwedge_{B\in 2^Y}(\mathrm{B}_Y(B)\to \mathrm{B}_X(f^{-1}(B))).$$

4.2 Remark If $BP(f)=T_M$, by Definition 2.3, we know $\mathrm{B}_X(f^{-1}(B))\geq \mathrm{B}_Y(B)$ for all $B\in 2^Y$. This is exactly the definition of bounded-preserving mappings in *M*-fuzzifying bornological space.

4.3 Proposition Let $f:(X,\mathrm{B}_X)\to(Y,\mathrm{B}_Y)$ and $g:(Y,\mathrm{B}_Y)\to(Z,\mathrm{B}_Z)$ be mappings between *M*-fuzzifying bornological spaces. Then

$$BP(f)\wedge BP(g)\leq BP(g\circ f).$$

Proof. By Definition 4.1, we have

$$\begin{aligned}
&BP(f)\wedge BP(g)\\
&=\bigwedge_{A\in 2^Y}\left(\mathrm{B}_Y(A)\to \mathrm{B}_X(f^{-1}(A))\right)\wedge\bigwedge_{B\in 2^Z}\left(\mathrm{B}_Z(B)\to \mathrm{B}_Y(g^{-1}(B))\right)\\
&\leq\bigwedge_{C\in 2^Z}\left(\mathrm{B}_Y(g^{-1}(C))\to \mathrm{B}_X(f^{-1}(g^{-1}(C)))\right)\wedge\bigwedge_{B\in 2^Z}\left(\mathrm{B}_Z(B)\to \mathrm{B}_Y(g^{-1}(B))\right)\\
&=\bigwedge_{B\in 2^Z}\left(\mathrm{B}_Y(g^{-1}(B))\to \mathrm{B}_X(f^{-1}(g^{-1}(B)))\right)\wedge\left(\mathrm{B}_Z(B)\to \mathrm{B}_Y(g^{-1}(B))\right)\\
&\leq\bigwedge_{B\in 2^Z}\left(\mathrm{B}_Z(B)\to \mathrm{B}_X((g\circ f)^{-1}(B))\right)\\
&=BP(g\circ f).
\end{aligned}$$

4.4 Corollary Let $f:(X,\mathrm{B}_X)\to(Y,\mathrm{B}_Y)$ and $g:(Y,\mathrm{B}_Y)\to(Z,\mathrm{B}_Z)$ be mappings between *M*-fuzzifying bornological spaces.

(3) If *f* is surjective, then $BP(g\circ f)\wedge BP(f)\leq BP(g)$;

(4) If *f* is injective, then $BP(g\circ f)\wedge BP(g)\leq BP(f)$.

4.5 Proposition Let (X,B) be an *M*-fuzzifying bornological space, $(Y,\mathrm{B}|_Y)$ be the subspace of (X,B), and $f:X\to Y$ be the including mapping. Then $BP(f)=T_M$.

Proof. By Definition 2.6, we have

$$\begin{aligned}
BP(f)&=\bigwedge_{A\in 2^Y}(\mathrm{B}|_Y(A)\to \mathrm{B}(A))\\
&=\bigwedge_{A\in 2^Y}\left(\left(\bigvee_{B\in 2^X, B|_Y=A}\mathrm{B}(B)\right)\to \mathrm{B}(A)\right)\\
&=\bigwedge_{A\in 2^Y}\ \bigwedge_{B\in 2^X, B|_Y=A}\left(\mathrm{B}(B)\to \mathrm{B}(A)\right)\\
&=T_M.
\end{aligned}$$

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